USE OF INVERSION TRANSFORMATION FOR MODELING IN CERTAIN PROBLEMS OF THE THEORY OF ELASTICITY AND PLASTICITY

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UDC 539.3

Certain possibilities of using the inversion transformation for modeling problems of the theory of elasticity and plasticity are considered, in particular, for cases where this allows us to make the experimental investigation on the model considerably easier to carry out.

The invariance of the biharmonic equation relative to inversion [1] was investigated by Michell [2] in the analysis of the first fundamental problem of the plane theory of elasticity. He showed that the inversion transformation "translates" any such problem for any particular body into another problem of the same physical type, but already for another body with another load. Here it is very important that the quantities which for the solution of the new problem must be known – the contour of the body and the conditions on the contour (i.e., loads) – are simply expressed in terms of the quantities specified when solving the original problem (loads, with accuracy up to the hydrostatic compression). Therefore inversion can be used for modeling. The situation in the problem concerned with the static flexure of a slab is analogous [3, 4].

Here we note that inversion can be used for modeling certain other problems of the theory of elasticity and plasticity which reduce to a biharmonic equation with a right side, and namely for modeling the plane thermoelastic stationary problem, when heat emission is specified in the body and the contour of the body is free from fixing; it can also be used for the problem concerned with the dynamic flexure of a homogeneous elastic slab resting on an inhomogeneous elastic-plastic foundation of the Winkler type. The mass of the slab is an arbitrary function of the point. Conditions of the "kinematic type" are specified on the contour.

In addition, inversion can be used in the case of the plane first fundamental isothermal problem for a linearly viscoelastic body and the case of the problem concerned with the dynamic flexure of a slab on a solid foundation, when the material of the slab possesses linear viscoelasticity and the material of the foundation has arbitrary nonlinear viscoelasticity. It can be used in these two problems when the material possesses circular orthotropy (the equation is no longer biharmonic) as well as in certain other problems.

In all problems the region is considered to be finite, while in the case of plane problems, for the sake of simplicity, it is assumed to be simply connected. In other respects the outline of the body or slab as well as the boundary conditions (in the framework of the types being considered) can be arbitrary. We assume that the center of the inversion circle lies outside the region occupied by the body. In the case of circular orthotropy it coincides with the center of orthotropy.

In the orthonormed x, y coordinate system the thermoelastic problem and the problem of slab flexure are described by the equations

$$\nabla^4\varphi = \frac{\alpha E}{\lambda(1-\mu)}e \tag{1}$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 134-136, March-April, 1972. Original article submitted October 1, 1971.

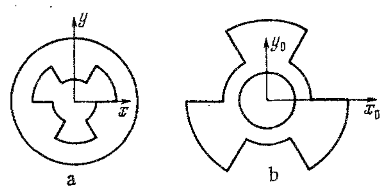

Fig. 1

$$D\nabla^4 w = q - \rho \frac{\partial^2 w}{\partial t^2} + p\,[w(t^*)]_{t^*=0}^{t^*=t} \tag{2}$$

Here φ is the Airy function; α, E, λ, μ are constants; e(x, y) is the specified intensity of heat emission per unit volume; D is the rigidity of the slab; w is the deflection; q (x, y, t) is the transverse load (a function to be given); ρ(x, y) is the mass per unit area of the middle surface (a function to be given); p is the reactive load acting on the slab from the side of the foundation; and $p[\ ]_{t^*=0}^{t^*=t}$ is the operator placing the function p(t) into correspondence with the function w(t). This operator characterizes the elastic-plastic properties of the foundation and is a given function of x, y. From the foundation we demand only that it can be described by a "point" operator. Thus, (2) describes the flexure of the slab on an elastic-plastic foundation of a very arbitrary type. In the case of a linearly elastic foundation p = c(x, y) w, where c(x, y) is a negative "constant." We note that the operator p[ ] can describe not only the traditional solid foundation – the ground – but also either damping effect of the material of the slab itself or the "point" resistance of any external medium.

The inverse transform

$$x = x_0/(x_0^2 + y_0^2),\quad y = y_0/(x_0^2 + y_0^2) \tag{3}$$

can be used to model the problems (1) and (2). Indeed, substituting in (1) and (2) instead of x and y their expressions in terms of x_0 and y_0 according to (3) and instead of φ and w

$$\varphi(x, y) = \varphi_0(x_0, y_0)/(x_0^2 + y_0^2),\qquad w(x, y, t) = w_0(x_0, y_0, t)/(x_0^2 + y_0^2) \tag{4}$$

we arrive at equations of the previous physical type but already with the "new" e_0 (x_0, y_0), q_0 (x_0, y_0, t), ρ_0 (x_0, y_0), $p_0[\ ]$. At the corresponding points

$$e_0 = e/(x_0^2 + y_0^2)^3 \tag{5}$$

$$q_0 = q/(x_0^2 + y_0^2)^3 \tag{6}$$

$$\rho_0 = \rho/(x_0^2 + y_0^2)^4 \tag{7}$$

$$p_0\,[w_0(t^*)]_{t^*=0}^{t^*=t} = \frac{1}{(x_0^2 + y_0^2)^3}\, p\left[\frac{w_0(t^*)}{x_0^2 + y_0^2}\right]_{t^*=0}^{t^*=t} \tag{8}$$

The boundary and initial conditions of the new problems are simply expressed in terms of the conditions of the original problems (in the case of the plane problem, with accuracy up to the hydrostatic compression).

Thus, if we are given a problem of type (1) or (2), then on the basis of (3)-(8) we can choose any of the "inversion" bodies, obtain the solution (mathematical, experimental, or mixed), and translate the results to the original body. In particular cases a body with nonuniform heat emission or a plate with a nonuniform mass and inhomogeneous foundation can be reduced to models with the constant e_0, ρ_0, and $p_0[\ ]$. Let, for example, it be necessary to determine the natural frequencies and natural modes of vibration of a slab (Fig. 1a) with $\rho = 1/(x^2+y^2)^4$. After inversion we arrive at the problem for a plate (Fig. 1b) with ρ_0 = const, on which it is easy to carry out an experiment. Alternatively, let the slab in Fig. 1a have an inhomogeneous nonlinearly elastic foundation with the characteristic

$$p(w) = \frac{1}{(x^2 + y^2)^3}\, f\left(\frac{w}{x^2 + y^2}\right)$$

where $f(\ )$ is an arbitrary function of its argument. After inversion we arrive at the problem for the slab (Fig. 1b) with a homogeneous foundation $p_0(w_0) = f(w_0)$. In the case of viscoelasticity the proof is analogous.

The fact mentioned in the paper that inversion can be used to model the problems (1) and (2) is not trivial, since there are a multitude of problems with equations which are close to (1) and (2), which when inverted "lose" their physical type (for example, the problem of slab stability) and a multitude of problems which, although "preserving" their physical type, have boundary conditions that are not transformed into one another in the inversion transformation (for example, the same problem (2), but with "solid" boundary conditions). In the latter it is also impossible to use inversion for modeling since up to the solution of the original problem we do not know the boundary conditions in the model.

We consider the case of materials possessing circular orthotropy. The contour of the body or slab and the boundary conditions can, let it be understood, possess no circular symmetry. Since the equations are of the same type, we write only the equation of plate flexure [5]:

$$
\begin{aligned}
&D_{r} \frac{\partial^4 w}{\partial r^4} + 2D_{r\theta} \frac{1}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} + D_{\theta} \frac{1}{r^4} \frac{\partial^4 w}{\partial \theta^4} + 2D_{r} \frac{1}{r} \frac{\partial^3 w}{\partial r^3} - 2D_{r\theta} \frac{1}{r^3} \frac{\partial^3 w}{\partial r \partial \theta^2} - \\
&- D_{\theta} \frac{1}{r^2} \frac{\partial^2 \omega}{\partial r^2} + 2(D_{\theta} + D_{r\theta}) \frac{1}{r^4} \frac{\partial^2 w}{\partial \theta^2} + D_{\theta} \frac{1}{r^3} \frac{\partial w}{\partial r} = q - \rho \frac{\partial^2 w}{\partial t^2} + p\,[w(t^*)]_{t^*=0}^{t^*=t}
\end{aligned}
\qquad (9)
$$

Here r, θ are the polar coordinates, D_r, D_θ, and $D_{r\theta}$ are other constants written in terms of local orthonormed bases. The transformation (3), (4) does not change the form of Eq. (9) [the right side is altered according to (6)-(8)]. We note that an equation of the type (9) would be invariant relative to (3), (4) also in the case where its left side contains not three but five independent constant coefficients (for all fourth and second derivatives).

The boundary conditions in the problems being considered, coinciding with the conditions in the case of isotropic bodies, are simply transformed. Therefore inversion can be used for modeling.

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